MATH5633 Loss Models I Autumn 2024

Chapter 4: Aggregate Risk Models

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Preview

In this chapter, we will look at aggregate risk models, which combine claim frequency and severity. The main focus will be on collective risk models, where the aggregate loss is modelled by a compound distribution. Different methods in computing aggregate loss distributions will be introduced. We will also look at the impact of individual and group deductibles.

Key topics in this chapter:

1. Individual and collective risk models;
2. Panjer's recursion;
3. Approximation methods for aggregate loss;
4. Coverage modifications in severity and aggregate loss.

1 Individual Risk Model

Consider a portfolio with n individual insurance policies. Let X_i, $i = 1, 2, \ldots, n$ be the severity of the i-th policy. Then, the aggregate loss S_n is given by

$$\boxed{S_n = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i.}$$

In this course, we will assume that $X_1, \ldots, X_n$ being (mutually) independent. The relationship of distributional quantities between S and the severity distributions are listed below.

1. **Mean and Variance**:

$$\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] \quad \text{and} \quad \mathrm{Var}[S_n] = \sum_{i=1}^{n} \mathrm{Var}[X_i].$$

2. **Generating Functions**:

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) \quad \text{and} \quad P_{S_n}(t) = \prod_{i=1}^{n} P_{X_i}(t).$$

1.1 Convolutions

Assume that $X_1, \ldots, X_n$ are i.i.d. variables with a common distribution X. The distribution of the aggregate loss S_n is given by the ***n-fold convolution*** of the distribution X.

Definition 1.1 Let $X_1, \ldots, X_n$ be i.i.d. variables with a common distribution X. The cdf of S is given by $F_S(s) = F_X^{*n}(s)$, where

$$\boxed{F_X^{*n}(s) = \mathbb{P}(X_1 + \cdots + X_n \leq s), \;\; s \in \mathbb{R}.} \tag{1}$$

It is clear that for $n = 1$, $F_X^{*1}(s) = \mathbb{P}(X_1 \leq s) = F_X(s)$. For $n \geq 1$, we have the following recursive formula:

Proposition 1.1 Let X be a non-negative random variable. For $n \geq 1$, F_X^{*n} can be computed recursively by

$$\boxed{F_X^{*(n+1)}(s) = \begin{cases} \displaystyle\sum_{x \leq s} F_X^{*n}(s-x) p_X(x), & \text{if } X \text{ is discrete;} \\ \displaystyle\int_0^s F_X^{*n}(s-x) f_X(x) dx, & \text{if } X \text{ is continuous.} \end{cases}}$$

Proof. We only consider the case when X is continuous. For any $n \geq 1$, by the law of total probability,

$$\begin{aligned} F^{*(n+1)}(s) &= \mathbb{P}(X_1 + \cdots + X_n + X_{n+1} \leq s) = \mathbb{P}(S_n \leq s - X_{n+1}) \\ &= \int_0^s \mathbb{P}(S_n \leq s - x | X_{n+1} = x) f_X(x) dx \\ &= \int_0^s \mathbb{P}(S_n \leq s - x) f_X(x) dx \\ &= \int_0^s F_X^{*n}(s-x) f_X(x) dx, \end{aligned}$$

where the third line follows from the independence of S_n and X_{n+1}. Notice that the integrating region is from 0 to s, since $S_{n+1} \leq s$ implies $X_{n+1} \leq s$. □

For some special severity distributions, even $X_1, \ldots, X_n$ are not identically distributed, the distribution of the aggregate loss S_n will belong to the same type of distribution, see Table 1 below.

X_i	S_n
$\mathcal{N}(\mu_i, \sigma_i^2)$	$\mathcal{N}(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
$\text{Exp}(\theta)$	$\text{Gamma}(n, \theta)$
$\text{Gamma}(\alpha_i, \theta)$	$\text{Gamma}(\sum_{i=1}^n \alpha_i, \theta)$

Table 1: Relationship between distributions of severity and aggregate losses

Example 1.1 An insurance portfolio consists of 3 policies. Suppose that the severity of the 3 policies are i.i.d. exponential distribution with a common mean $1,200$. Find the probability that the aggregate loss is at least $5,000$.

Solution:

Since $X_1, X_2, X_3 \sim \text{Exp}(1200)$, we have $S_3 \sim \text{Gamma}(3, 1200)$. The pdf of S_3 is given by

$$f_{S_3}(s) = \frac{1}{2(1200)^3} s^2 e^{-\frac{s}{1200}}, \ \ s > 0.$$

Hence,

$$\mathbb{P}(S_3 > 5000) = \int_{5000}^{\infty} \frac{1}{2(1200)^3} s^2 e^{-\frac{s}{1200}} ds = 0.2147.$$

2 Collective Risk Model

In practice, the number of claims N is random. If $X_1, X_2, \ldots,$ are the severity of individual claims, the aggregate loss S can be modelled by the following *compound variable*:

$$\boxed{S = X_1 + X_2 + \cdots + X_N = \sum_{i=1}^{N} X_i.}$$

We shall make the following assumptions:

1. When $N = 0$, $S := 0$;
2. $X_1, X_2, \ldots,$ are i.i.d. with a common distribution X;
3. $X_1, X_2, \ldots,$ are independent of N, i.e., individual claim sizes and number of claims are independent.

2.1 Mean and Variance of Aggregate Loss

From Section 5 of Chapter 3, we know that S follows a *compound distribution*, where the claim frequency N is the *primary distribution*, and the loss severity X is the *secondary distribution*. By Theorem 5.1 and Proposition 5.2 in Chapter 3, we also have the following:

1. **Mean and Variance**:

$$\mathbb{E}[S] = \mathbb{E}[X]\mathbb{E}[N] \quad \text{and} \quad \text{Var}[S] = \mathbb{E}[N]\text{Var}[X] + \mathbb{E}^2[X]\text{Var}[N].$$

2. **Generating Functions**:

$$P_S(t) = P_N(P_X(t)) \quad \text{and} \quad M_S(t) = P_N(M_X(t)).$$

Example 2.1 Suppose the severity of each claim are independent and identically distributed, which follows a common gamma distribution with $\alpha = 2$ and $\theta = 1,000$. Also, the claim frequency follows a Poisson distribution with mean 20. Calculate the expected value and the variance of the aggregate loss.

Solution:

Since $X \sim \text{Gamma}(2, 1000)$ and $N \sim \text{Poi}(20)$, we have $\mathbb{E}[X] = 2(1000) = 2000$, $\mathbb{E}[N] = 20$, $\text{Var}[X] = 2(1000)^2$ and $\text{Var}[N] = 20$. Hence,

$$\begin{aligned}\mathbb{E}[S] &= \mathbb{E}[X]\mathbb{E}[N] = 20(2000) = 40,000, \\ \text{Var}[S] &= \mathbb{E}[N]\text{Var}[X] + \mathbb{E}^2[X]\text{Var}[N] \\ &= 20 \times 2(1000)^2 + (2000^2)(20) \\ &= 60,000,000.\end{aligned}$$

Example 2.2 Suppose the distributions of the severity X of each claim, and the number of claims N, are discretely distributed with the following pmfs:

n	$\mathbb{P}(N = n)$	x	$\mathbb{P}(X = x)$
1	0.7	0	0.6
2	0.2	100	0.2
3	0.1	1000	0.2

A policy is written on the aggregate loss with a pure premium that equals the expected aggregate loss, plus 0.5 times of the standard deviation of the aggregate loss. Calculate the pure premium.

Solution:

From the table, we know that

$$\mathbb{E}[N] = 1.4,\ \mathbb{E}[N^2] = 2.4,\ \mathrm{Var}[N] = \mathbb{E}[N^2] - \mathbb{E}^2[N] = 0.44,$$

$$\mathbb{E}[X] = 220,\ \mathbb{E}[X^2] = 202,000,\ \mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X] = 153,600.$$

Hence,

$$\begin{aligned}\mathbb{E}[S] &= \mathbb{E}[X]\mathbb{E}[N] = 220(1.4) = 308,\\ \mathrm{Var}[S] &= \mathbb{E}[N]\mathrm{Var}[X] + \mathbb{E}^2[X]\mathrm{Var}[N]\\ &= (1.4)(153,600) + (220^2)(0.44)\\ &= 236,336.\end{aligned}$$

Therefore, the pure premium is given by

$$P = \mathbb{E}[S] + 0.5\sqrt{\mathrm{Var}[S]} = 308 + 0.5\sqrt{236,336} = 551.0720.$$

2.2 Distribution of Aggregate Loss

To deduce the cdf/pdf of the aggregate loss, we apply the law of total probability by conditioning on the number of claims N:

$$\begin{aligned}F_S(s) &= \mathbb{P}(X_1 + \cdots + X_N \le s)\\ &= \sum_{n=0}^{\infty} \mathbb{P}(X_1 + \cdots + X_n \le s | N = n)\mathbb{P}(N = n)\\ &= \sum_{n=0}^{\infty} F_X^{*n}(s)\mathbb{P}(N = n),\end{aligned}$$

where F_X^{*n} is the n-fold convolution of F_X defined in (1), see also Proposition 1.1. The pdf of S can thus obtained by differentiating the cdf:

$$f_S(s) = \sum_{n=0}^{\infty} f_X^{*n}(s)\mathbb{P}(N = n),$$

where $f_X^{*n}(s) = \frac{d}{ds} f_X^{*n}(s)$ is the cdf of $X_1 + \cdots + X_n$.

It is in general a difficult task to compute the cdf/pdf of the aggregate loss distribution, unless the severity and the frequency distributions take very special forms. In Sections 3-4, we will discuss some recursive and numerical methods to compute the distribution of S.

3 Convolution and Panjer's Recursion

In this section, we will deviate from the usual treatment, and assume the severity X follows a discrete distribution with support $\mathbb{N}_0$. Indeed, a continuous distribution can be approximated by discrete ones via *discretization*; see Section 4.1. By this assumption, the distributions of X, N, and S are all discrete. We adopt the following notation: for $n = 0, 1, 2, \ldots,$

1. $p_n := \mathbb{P}(N = n)$;
2. $f_n := \mathbb{P}(X = n)$;
3. $g_n := \mathbb{P}(S = n)$.

By the convolution formula, the pmf $\{g_n\}_{n=0}^{\infty}$ of S can be computed by

$$\boxed{g_k = \sum_{j=0}^{\infty} p_j \sum_{i_1+\cdots+i_j=k} f_{i_1} \cdots f_{i_j} = \sum_{j=0}^{\infty} p_j f_k^{*j},} \tag{2}$$

where

$$f_k^{*j} := \mathbb{P}(X_1 + \cdots + X_j = k) = \sum_{i_1+\cdots+i_j=k} f_{i_1} \cdots f_{i_j}.$$

Example 3.1 For an insurance coverage, the number of claims follows a Poisson distribution with mean 5. Claim size is independent of the claim frequency, which is distributed as follows:

Claim Size	Probability
1	0.3
2	0.5
3	0.2

Let S be the aggregate loss. Calculate $\mathbb{P}(S \le 3)$.

<u>*Solution*</u>:
We compute g_0, g_1, g_2, and g_3 by the convolution formula. First, $S = 0$ iff $N = 0$. Hence $g_0 = p_0 = e^{-5}$.
$S = 1$ iff $N = 1$ and $X = 1$. Hence,

$$g_1 = p_1 f_1 = 0.3\left(5e^{-5}\right) = 1.5e^{-5}.$$

Next, $S = 2$ if $N = 1$ and $X = 2$, or $N = 2$ and $X_1 = X_2 = 1$. Hence,

$$g_2 = p_1 f_2 + p_2 f_1^2 = 0.5(5e^{-5}) + 0.3^2\left(\frac{25e^{-5}}{2}\right) = 3.625e^{-5}.$$

Finally, $S = 3$ if $N = 1$ and $X = 3$, or $N = 2$ with loss sizes 1 and 2 (with 2 permutations), or $N = 3$ and each loss has a loss size of 1. Hence,

$$g_3 = p_1 f_3 + 2p_2 f_1 f_2 + p_3 f_1^3 = 0.2(5e^{-5}) + 2(12.5e^{-5})(0.3)(0.5) + (0.3)^3 \left(\frac{125e^{-5}}{6}\right)$$
$$= 5.3125e^{-5}.$$

Therefore,

$$\mathbb{P}(S \le 3) = (1 + 1.5 + 3.625 + 5.3125)\, e^{-5} = 11.4375e^{-5} = 0.0771.$$

When the frequency distribution N belongs to the $(a, b, 0)$ or the $(a, b, 1)$ class, the pmf of S can be computed using simplier recursive formulas, known as ***Panjer's recursion***:

Theorem 3.1 Suppose that N belongs to the $(a, b, 0)$ class. Then, the pmf $\{g_k\}_{k=0}^{\infty}$ of the aggregate loss S satisfies the following recursion:

$$\boxed{\begin{aligned} &g_0 = P_N(P_X(0)) = P_N(f_0), \\ &g_k = \frac{1}{1 - af_0} \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) f_j g_{k-j}, \; k = 1, 2, \ldots \end{aligned}} \tag{3}$$

The recursion (3) allows us to compute the pmf of S without having to explicitly compute the convolution f_n^{*j}. During exams, you will be asked to compute g_k only for small values of k. Below we provide a proof for the case $N \sim \text{Poi}(\lambda)$, i.e., $a = 0$ and $b = \lambda$.

Proof. Consider the pgf of S, which is given by

$$P_S(t) = g_0 + \sum_{k=1}^{\infty} g_k t^k. \tag{4}$$

Differentiating both sides with respect to t yields

$$P_S'(t) = \sum_{k=1}^{\infty} k g_k t^{k-1}. \tag{5}$$

On the other hand, when $N \sim \text{Poi}(\lambda)$, we can also write the pgf of S as

$$P_S(t) = P_N(P_X(t)) = e^{\lambda(P_X(t)-1)}.$$

By differentiating both sides with respect to t, we obtain

$$P_S'(t) = \lambda e^{\lambda(P_X(t)-1)} P_X'(t) = \lambda P_X'(t) P_S(t). \tag{6}$$

Substituting (4) and

$$P_X'(t) = \sum_{j=1}^{\infty} j f_j t^{j-1},$$

into (6), we have

$$\begin{aligned} P_S'(t) &= \lambda \left(\sum_{j=1}^{\infty} j f_j t^{j-1} \right) \left(g_0 + \sum_{i=1}^{\infty} g_i t^i \right) \\ &= \lambda \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} j f_j g_{k-j} \right) t^{k-1}. \end{aligned} \tag{7}$$

By matching the coefficients of t^{k-1} of (5) and (7), we obtain

$$g_k = \sum_{j=1}^{k} \frac{\lambda j}{k} f_j g_{k-j},$$

which agrees with (3) for $a = 0$ and $b = \lambda$.

□

When N belongs to the $(a, b, 1)$ class, we have the following Panjer's recursion:

Theorem 3.2 Suppose that N belongs to the $(a, b, 1)$ class. Then, the pmf $\{g_k\}_{k=0}^{\infty}$ of the aggregate loss S satisfies the following recursion:

$$\boxed{\begin{aligned} &g_0 = P_N(P_X(0)) = P_N(f_0), \\ &g_k = \frac{1}{1 - a f_0} \left[(p_1 - (a+b)p_0)\, f_k + \sum_{j=1}^{k} \left(a + \frac{bj}{k} \right) f_j g_{k-j} \right], \quad k = 1, 2, \ldots \end{aligned}} \tag{8}$$

Example 3.2 Repeat Example 3.1 using Panjer's recursion.

Solution:

$N \sim \text{Poi}(5)$ belongs to the $(a, b, 0)$ class with $a = 0$ and $b = 5$. The mgf of N is $P_N(t) = e^{5(t-1)}$. Hence, $g_0 = P_N(f_0) = P_N(0) = e^{-5}$. Using Panjer's recursion, we have

$$\begin{aligned} g_1 &= \frac{5 \cdot 1}{1} f_1 g_0 = 5(0.3)(e^{-5}) = 1.5e^{-5}, \\ g_2 &= \frac{5 \cdot 1}{2} f_1 g_1 + \frac{5 \cdot 2}{2} f_2 g_0 = 2.5(0.3)(1.5e^{-5}) + 5(0.5)(e^{-5}) = 3.625e^{-5}, \\ g_3 &= \frac{5 \cdot 1}{3} f_1 g_2 + \frac{5 \cdot 2}{3} f_2 g_1 + \frac{5 \cdot 3}{3} f_3 g_0 = \frac{5}{3}(0.3)(3.625e^{-5}) + \frac{10}{3}(0.5)(1.5e^{-5}) + 5(0.2)(e^{-5}) \\ &= 5.3125e^{-5}. \end{aligned}$$

Hence, we get the same g_0, g_1, g_2, and g_3, and thus $\mathbb{P}(S \leq 3) = 0.0771$.

4 Approximation Methods

Computing the distribution of the compound variable S is in general difficult. In this section, we discuss two approximation methods to compute the distribution of S.

4.1 Method of Rounding

Panjer's recursion is only applicable when X follows a discrete distribution, which is not a very realistic assumption. Nevertheless, even X is continuously distributed, we can estimate it using a discrete distribution by ***discretization***. The ***method of rounding*** is a special technique to discretize a continuous distribution.

Let X be a continuous, non-negative random variable. We approximate X by a discrete random variable X', which is supported in $\{0, h, 2h, 3h, \cdots\} = \{kh : k \in \mathbb{N}_0\}$. We call the value $h > 0$ the ***span***. The pmf $\{f_{kh}\}_{k=0}^{\infty}$ of X' is given by the following:

$$\boxed{\begin{aligned} f_0 &= \mathbb{P}(X' = 0) = F_X(0.5h), \\ f_{kh} &= \mathbb{P}(X' = kh) = F_X\left((k+0.5)h\right) - F_X\left((k-0.5)h\right), \ k = 1, 2, 3, \ldots. \end{aligned}}$$

Indeed, for $k \in \mathbb{N}$, the probability f_{kh} is given by

$$f_{kh} = \int_{(k-0.5)h}^{(k+0.5)h} f_X(x)dx,$$

which roughly approximates the probability of X around kh.

Example 4.1 Claim counts follow a Poisson distribution with mean 3. Claim sizes follow an exponential distribution with mean 2. Claim counts and claim sizes are independent. The severity distribution is discretized using the method of rounding with span 1. Using the discrete approximation of the severity distribution, calculate the probability that the aggregate loss is less than or equal to 3.

Solution:

The cdf of X is given by $F_X(x) = 1 - e^{-x/2}$. Using this, we can compute the pmf of X':

$$\begin{aligned} f_0 &= F_X(0.5) = 1 - e^{-\frac{0.5}{2}} = 0.2212, \\ f_1 &= F_X(1.5) - F_X(0.5) = e^{-\frac{0.5}{2}} - e^{-\frac{1.5}{2}} = 0.3064, \\ f_2 &= F_X(2.5) - F_X(1.5) = e^{-\frac{1.5}{2}} - e^{-\frac{2.5}{2}} = 0.1859, \\ f_3 &= F_X(3.5) - F_X(2.5) = e^{-\frac{2.5}{2}} - e^{-\frac{3.5}{2}} = 0.1127. \end{aligned}$$

Since $N \sim \text{Poi}(3)$, we have $a = 0$ and $b = 3$. Hence,

$$g_0 = P_N(f_0) = e^{3(f_0-1)} = 0.0967.$$

Using Panjer's recursion, we have

$$\begin{aligned}
g_1 &= \frac{3 \cdot 1}{1} f_1 g_0 = 3(0.3064)(0.0967) = 0.0889, \\
g_2 &= \frac{3 \cdot 1}{2} f_1 g_1 + \frac{3 \cdot 2}{2} f_2 g_0 = 1.5(0.3064)(0.0889) + 3(0.1859)(0.0967) = 0.0948, \\
g_3 &= \frac{3 \cdot 1}{3} f_1 g_2 + \frac{3 \cdot 2}{3} f_2 g_1 + \frac{3 \cdot 3}{3} f_3 g_0 \\
&= (0.3064)(0.0948) + 2(0.1859)(0.0889) + 3(0.1127)(0.0967) = 0.0948.
\end{aligned}$$

Therefore, the required probability is $g_0 + g_1 + g_2 + g_3 = 0.3751$.

4.2 Normal Approximation

When $\mathbb{E}[N]$ is sufficiently large, the *normal approximation* could be a convenient method to compute the distribution of S, as justified by the *central limit theorem*:

$$\boxed{\frac{S - \mathbb{E}[S]}{\sqrt{\text{Var}[S]}} \xrightarrow{d} \mathcal{N}(0.1).}$$

If you were asked to compute $\mathbb{P}(S > a)$, you will need to follow the steps below:

1. Compute $\mathbb{E}[S]$ and $\text{Var}[S]$;
2. Continuity corrections for discrete distributions: suppose that S could take values $\{s_0, s_1, s_2, \cdots\}$ with $s_{n+1} > s_n$ for $n \geq 0$, the following adjustments have to be made:

Probability	Corrections
$\mathbb{P}(S > s_n)$ $\mathbb{P}(S \geq s_{n+1})$ $\mathbb{P}(S > s)$, $s \in (s_n, s_{n+1})$	$\mathbb{P}\left(S > \frac{s_n+s_{n+1}}{2}\right)$
$\mathbb{P}(S \leq s_n)$ $\mathbb{P}(S < s_{n+1})$ $\mathbb{P}(S < s)$, $s \in (s_n, s_{n+1})$	$\mathbb{P}\left(S < \frac{s_n+s_{n+1}}{2}\right)$

Table 2: Continuity corrections for discrete severity

Example 4.2 You are given the following:

- A portfolio consists of 1,600 independent risks.
- For each risk, the probability of at least one claim is 0.4.

Using the Central Limit Theorem, determine the approximate probability that the number of risks in the portfolio with at least one claim will be greater than 620.

Solution:

The distribution of the number of risks N in the portfolio with at least one claim is given by $N \sim \text{Bin}(1600, 0.4)$. Hence, $\mathbb{E}[N] = 1600(0.4) = 640$ and $\text{Var}[N] = 1600(0.4)(1 - 0.4) = 384$. Let $Z \sim \mathcal{N}(0,1)$. Since N is discretely distributed, by continuity correction,

$$\mathbb{P}(N > 620) \approx \mathbb{P}\left(Z > \frac{620.5 - 640}{\sqrt{384}}\right) = 1 - \Phi(-0.9951) = 0.8402.$$

Example 4.3 (SOA 2018 STAM SAMPLE QUESTION 91) The number of auto vandalism claims reported per month at Sunny Daze Insurance Company (SDIC) has mean 110 and variance 750. Individual losses have mean 1101 and standard deviation 70. The number of claims and the amounts of individual losses are independent. Using the normal approximation, calculate the probability that SDIC's aggregate auto vandalism losses reported for a month will be less than 100,000.

Solution:

We first calculate the mean and variance of the aggregate loss:

$$\begin{aligned}\mathbb{E}[S] &= \mathbb{E}[X]\mathbb{E}[N] = (110)(1101) = 121,110, \\ \text{Var}[S] &= \mathbb{E}[N]\text{Var}[X] + \mathbb{E}^2[X]\text{Var}[N] = (110)(70^2) + (1101)^2(750) = 909,689,750.\end{aligned}$$

By normal approximation, we have

$$\mathbb{P}(S < 100,000) \approx \mathbb{P}\left(Z < \frac{100,000 - 121,110}{\sqrt{909,689,750}}\right) = \mathbb{P}(Z < -0.6999) = 0.2420.$$

5 Aggregate Payment with Severity Deductibles

The variable S computes the aggregate ground-up losses of all issued policies. However, when (ordinary or franchise) deductibles are imposed on the policies, not all losses are going to lead to payments. Below we introduce two methods to compute the aggregate payments.

- **Method 1: Per-loss basis**

  Let d be an ordinary deductible applied to each policy, and N^L be the number of loss.

Then, the payment from the k-th loss, $k = 1, 2, \dots,$ is given by $Y_k^L := (X_k - d)_+$. The ***aggregate payment*** is thus

$$\boxed{S^L = Y_1^L + Y_2^L + \cdots + Y_{N^L}^L = \sum_{k=1}^{N^L} Y_k^L.}$$

Using this, the *expected aggregate payments* and the *variance of aggregate payments* are given by

$$\boxed{\mathbb{E}[S^L] = \mathbb{E}[Y^L]\mathbb{E}[N^L] = \mathbb{E}[(X-d)_+]\mathbb{E}[N^L].}$$

- **Method 2: Per-payment basis**

  In the second method, instead of summing the payment variables for all claims, we only aggregate those that will lead to a payment (i.e., losses that exceed the deductible). Let N^P be the *number of payments.* Recall that

  $$N^P = I_1 + I_2 + \cdots + I_{N^L} = \sum_{k=1}^{N^L} I_k,$$

  where for each $k \geq 1$, $I_k \sim \text{Bernoulli}(v)$, and $v = \mathbb{P}(X > d)$; see Section 6 of Chapter 3 for details. Let $Y_k^P := X_k - d | X_k > d$ be the payment per payment variable. Then, the aggregate payment is given by

  $$\boxed{S^P = Y_1^P + Y_2^P + \cdots + Y_{N^P}^P = \sum_{k=1}^{N^P} Y_k^L.}$$

  Using this, the ***expected aggregate payment*** is given by

  $$\boxed{\mathbb{E}[S^P] = \mathbb{E}[Y^P]\mathbb{E}[N^P] = \mathbb{E}[X - d|X > d]\mathbb{E}[N^P].}$$

As shown by the following theorem, the two methods indeed give the same result.

Theorem 5.1 The variables S^L and S^P have the same distribution.

Proof. We shall show that the pgfs of S^L and S^P are the same. We first derive the pgf of S^P, where we will need the pgf of N^P. Let $v = \mathbb{P}(X > d)$ and $I \sim \text{Bernoulli}(v)$, we have

$$P_{N^P}(t) = P_{N^L}(P_I(t)) = P_{N^L}(1 - v + vt).$$

The pgf of S^P is thus given by

$$P_{S^P}(t) = P_{N^P}\left(P_{Y^P}(t)\right) = P_{N^L}(1 - v + vP_{Y^P}(t)).$$

Next, we derive the pgf of S^L, which is given by

$$P_{S^L}(t) = P_{N^L}(P_{Y^L}(t))$$

By conditioning on the event $Y^L > 0$ and $Y^L = 0$ (recall that $Y^L \geq 0$, so we either have $Y^L > 0$ or $Y^L = 0$), and noticing that $Y^L | Y^L > 0 = X - d | X > d = Y^P$, we have

$$\begin{aligned} P_{Y^L}(t) &= \mathbb{E}[t^{Y^L}] = \mathbb{E}[t^{Y^L} | Y^L > 0]\mathbb{P}(Y^L > 0) + \mathbb{E}[t^{Y^L} | Y^L = 0]\mathbb{P}(Y^L \leq 0) \\ &= \mathbb{E}[t^{Y^P}]v + (1)(1 - v) \\ &= 1 - v + vP_{Y^P}(t). \end{aligned}$$

Therefore,

$$P_{S^L}(t) = P_{N^L}(P_{Y^L}(t)) = P_{N^L}(1 - v + vP_{Y^P}(t)) = P_{S^P}(t).$$

□

Remark 5.2. When calculating the aggregate payment, either use the per-loss basis (using N^L and Y^L), or the per-payment basis (using N^P and Y^P), and do NOT mix up the two.

Example 5.1 (SOA 2018 STAM SAMPLE QUESTION 212) For an insurance:

(i) The number of losses per year has a Poisson distribution with $\lambda = 10$.
(ii) Loss amounts are uniformly distributed on (0, 10).
(iii) Loss amounts and the number of losses are mutually independent.
(iv) There is an ordinary deductible of 4 per loss.

Calculate the variance of aggregate payments in a year.

Solution:
We solve the problem using both the per-loss and the per-payment basis.
Per-loss basis:
For the per-loss basis, we have $\mathbb{E}[N^L] = 10 = \text{Var}[N^L]$. On the other hand,

$$\begin{aligned} \mathbb{E}[Y^L] &= \int_4^{10} (x - 4) f_X(x) dx = \int_4^{10} \frac{x - 4}{10} dx = 1.8, \\ \mathbb{E}[(Y^L)^2] &= \int_4^{10} \frac{(x - 4)^2}{10} dx = 7.2, \\ \text{Var}[Y^L] &= 7.2 - 1.8^2 = 3.96. \end{aligned}$$

Hence,

$$\text{Var}[S^L] = \mathbb{E}[N^L]\text{Var}[Y^L] + \mathbb{E}^2[Y^L]\text{Var}[N^L] = 10 \times 3.96 + 1.8^2 \times 10 = 72.$$

Per-payment basis:
Since $X \sim \text{Uniform}(0,10)$, we have $v = \mathbb{P}(X > 4) = 1 - 0.4 = 0.6$. Hence, $N^P \sim \text{Poi}(v\lambda) = \text{Poi}(6)$. On the other hand, the support of Y^P is $(0, 10-4) = (0,6)$, and $Y^P \sim \text{Uniform}(0,6)$. Indeed, for $y \in (0,6)$ the pdf of Y^P is given by

$$f_{Y^P}(y) = \frac{f_X(y+d)}{S_X(d)} = \frac{f_X(y+4)}{0.6} = \frac{1/10}{0.6} = \frac{1}{6}.$$

Using these, we have $\mathbb{E}[N^P] = 6 = \text{Var}[N^P]$, $\mathbb{E}[Y^P] = (0+6)/2 = 3$, $\text{Var}[Y^P] = (6-0)^2/12 = 3$. Therefore,

$$\text{Var}[S^P] = \mathbb{E}[N^P]\text{Var}[Y^P] + \mathbb{E}^2[Y^P]\text{Var}[N^P] = 6 \times 3 + 3^2 \times 6 = 72.$$

6 Deductibles on Aggregate Losses

In the last section, we discussed the aggregate payments when a deductible is imposed in individual losses. In this section, we consider the case when a deductible is imposed on the aggregate loss. For instance, reinsurance companies often cover the aggregate losses of an insurance company, applying a single deductible to the overall total.

Definition 6.1 Let S be the aggregate loss subject to a deductible d. This insurance is called ***stop-loss insurance***. The expected cost per loss of this insurance is called the ***(net) stop-loss premium***:

$$\boxed{\mathbb{E}[(S-d)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge d].}$$

If the distribution (pdf/pmf or cdf) of S is available, we can compute the stop-loss premium by the following formula:

$$\boxed{\begin{aligned}\mathbb{E}[(S-d)_+] &= \int_d^\infty (1 - F_S(s))ds \\ &= \begin{cases} \displaystyle\int_d^\infty (s-d) f_S(s)ds, & \text{if } S \text{ is continuous;} \\ \displaystyle\sum_{s>d} (s-d)p_S(s), & \text{if } S \text{ is discrete.}\end{cases}\end{aligned}}$$

Another way to compute the stop-loss premium is by utilizing the formula

$$\mathbb{E}[(S-d)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge d],$$

so we will have to compute $\mathbb{E}[S]$ and $\mathbb{E}[S \wedge d]$. The former is usually easy to compute (recall that $\mathbb{E}[S] = \mathbb{E}[X]\mathbb{E}[N]$), so we focus on the latter.

If S follows a discrete distribution, we can compute the pmf of S using methods in Section 3. We can then compute the stop-loss premium as follows:

Theorem 6.1 Suppose that S is a discrete random variable supported in $\{0, h, 2h, 3h, \dots\}$, where $h > 0$ is fixed. Then, for $d > 0$,

$$\boxed{\begin{aligned} \mathbb{E}[S \wedge d] &= h \sum_{m=0}^{u-1} [1 - F_S(mh)] + (d - uh)\,[1 - F_S(hu)] \\ \mathbb{E}[(S-d)_+] &= h \sum_{m=0}^{\infty} [1 - F_S((m+u)h)] - (d - uh)\,[1 - F_S(hu)]\,. \end{aligned}}$$

where $\boxed{u := \left\lceil \frac{d}{h} \right\rceil - 1}$.

Remark 6.2.

1. If $u - 1 < 0$, then $\sum_{m=0}^{u-1}[1 - F_S(mh)] = 0$, and the expected limited loss is reduced to

$$\mathbb{E}[S \wedge d] = (d - uh)[1 - F_S(uh)].$$

2. As a consequence of Theorem 6.1, if d is a multiple of h, then $u = d/h - 1$, and the formulas are reduced to

$$\boxed{\begin{aligned} \mathbb{E}[S \wedge d] &= h \sum_{m=0}^{d/h-1} [1 - F_S(mh)] \\ \mathbb{E}[(S-d)_+] &= h \sum_{m=0}^{\infty} [1 - F_S(mh + d)]\,. \end{aligned}}$$

3. Alternatively, using the pmf of S, we can compute $\mathbb{E}[S \wedge d]$ by

$$\boxed{\mathbb{E}[S \wedge d] = \sum_{m=0}^{u} mh\mathbb{P}(S = mh) + d\mathbb{P}(S > uh).}$$

Proof. By the definition of u, we have $uh < d \le (u+1)h$. Using this, we have

$$\mathbb{E}[S \wedge d] = \int_0^d [1 - F_S(s)]\,ds$$

$$
\begin{aligned}
&= \sum_{m=0}^{u-1} \int_{mh}^{(m+1)h} [1 - F_S(s)]\, ds + \int_{uh}^{d} [1 - F_S(s)]\, ds \\
&= \sum_{m=0}^{u-1} \int_{mh}^{(m+1)h} [1 - F_S(mh)]\, ds + \int_{uh}^{d} [1 - F_S(uh)]\, ds \\
&= h \sum_{m=0}^{u-1} [1 - F_S(mh)] + (d - uh)[1 - F_S(uh)].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathbb{E}[S] &= \int_0^{\infty} [1 - F_S(s)]\, ds \\
&= \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h} [1 - F_S(s)]\, ds \\
&= \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h} [1 - F_S(mh)]\, ds \\
&= h \sum_{m=0}^{\infty} [1 - F_S(mh)].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}[(S-d)_+] &= \mathbb{E}[S] - \mathbb{E}[S \wedge d] \\
&= h \sum_{m=0}^{\infty} [1 - F_S(mh)] - h \sum_{m=0}^{u-1} [1 - F_S(mh)] - (d - uh)[1 - F_S(uh)] \\
&= h \sum_{m=u}^{\infty} [1 - F_S(mh)] - (d - uh)[1 - F_S(uh)] \\
&= h \sum_{m=0}^{\infty} [1 - F_S((m+u)h)] - (d - uh)[1 - F_S(uh)].
\end{aligned}
$$

□

If we know the stop-loss premium when the deductible is a and b, and $\mathbb{P}(a < S < b) = 0$, we can then compute the stop-loss premium with deductible d for any $d \in [a, b]$ using interpolation.

Theorem 6.3 Suppose that $\mathbb{P}(a < S < b) = 0$. Then, for any $d \in [a, b]$,

$$
\boxed{\mathbb{E}[(S-d)_+] = \frac{b-d}{b-a}\mathbb{E}[(S-a)_+] + \frac{d-a}{b-a}\mathbb{E}[(S-b)_+].}
$$

Proof. If $\mathbb{P}(a < S < b) = 0$, then $\mathbb{E}[(S-d)_+]$ is a linear function of d when $d \in [a,b]$. Indeed,

$$\begin{aligned}\mathbb{E}[(S-d)_+] &= \int_d^\infty [1-F_S(s)]ds = \int_a^\infty [1-F_S(s)]ds - \int_a^d [1-F_S(s)]ds \\ &= \mathbb{E}[(S-a)_+] - (d-a)[1-F_S(a)].\end{aligned}$$

Since this holds for any $d \in [a,b]$, in particular, we have

$$\mathbb{E}[(S-b)_+] = \mathbb{E}[(S-a)_+] - (b-a)[1-F_S(a)].$$

This gives

$$1 - F_S(a) = \frac{\mathbb{E}[(S-a)_+] - \mathbb{E}[(S-b)_+]}{b-a}.$$

Substituting this into the first equation, we have

$$\begin{aligned}\mathbb{E}[(S-d)_+] &= \mathbb{E}[(S-a)_+] - (d-a)[1-F_S(a)] \\ &= \mathbb{E}[(S-a)_+] - (d-a)\left(\frac{\mathbb{E}[(S-a)_+] - \mathbb{E}[(S-b)_+]}{b-a}\right) \\ &= \frac{b-d}{b-a}\mathbb{E}[(S-a)_+] + \frac{d-a}{b-a}\mathbb{E}[(S-b)_+].\end{aligned}$$

□

Example 6.1 The number of claims of an insurance coverage has a geometric distribution with mean 4. The distribution of claim sizes is as follows:

x	$\mathbb{P}(X=x)$
2	0.45
4	0.25
6	0.2
8	0.1

A stop-loss reinsurance contract has a deductible of 5. Find the stop-loss premium for the reinsurance contract.

Solution:
Let S be the aggregate loss and N be the number of claims. We have $\mathbb{E}[X] = 2(0.45) + 4(0.25) + 6(0.2) + 8(0.1) = 3.9$. Hence, $\mathbb{E}[S] = \mathbb{E}[X]\mathbb{E}[N] = (3.9)(4) = 15.6$.
Next, we compute the distribution of S. Since X takes values in $2, 4, 6, 8$, S takes values in the set $\{0, 2, 4, 6, 8, \dots\}$, i.e., the set of positive even integers and zero. Using the notation in Theorem 6.1, we have $h = 2$ and $u = \lceil 5/2 \rceil - 1 = 2$. Next, we compute g_2, g_4, and g_6, where $g_k = \mathbb{P}(S = k)$. To this end, we first recall the pmf of N, which is given by

$$p_k = \mathbb{P}(N = k) = \left(\frac{4}{1+4}\right)^k \left(\frac{1}{1+4}\right) = 0.2(0.8)^k, \;\; k = 0, 1, 2 \dots.$$

Using this, we have $p_0 = 0.2$, $p_1 = (0.2)(0.8) = 0.16$, and $p_2 = (0.2)(0.8)^2 = 0.128$. Next, we compute the pmf of S:

$$
\begin{aligned}
g_0 &= \mathbb{P}(N=0) = p_0 = 0.2, \\
g_2 &= \mathbb{P}(N=1)\mathbb{P}(X=2) = (0.16)(0.45) = 0.072, \\
g_4 &= \mathbb{P}(N=1)\mathbb{P}(X=4) + \mathbb{P}(N=2)[\mathbb{P}(X=2)]^2 \\
&= (0.16)(0.25) + (0.128)(0.45)^2 = 0.06592, \\
\mathbb{P}(S>4) &= 1 - g_0 - g_2 - g_4 = 0.66208.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}[S \wedge 5] &= 0 \cdot g_0 + 2 \cdot g_2 + 4 \cdot g_4 + 5\mathbb{P}(S>5) \\
&= 2(0.072) + 4(0.06592) + 5\mathbb{P}(S>4) \\
&= 0.41368 + 3.2954 = 3.71808.
\end{aligned}
$$

Alternatively, using Theorem 6.3, by noticing that $\mathbb{P}(S>0) = 1 - g_0 = 0.8$, $\mathbb{P}(S>2) = 1 - g_0 - g_2 = 0.728$, we have

$$
\mathbb{E}[S \wedge 5] = 2\left[\mathbb{P}(S>0) + \mathbb{P}(S>2)\right] + (5-4)\mathbb{P}(S>4) = 3.71808.
$$

Therefore, $\mathbb{E}[(S-5)_+] = \mathbb{E}[S] - \mathbb{E}[S \wedge 5] = 15.6 - 3.71808 = 11.88192$.

Example 6.2 Repeat Example 6.1 using interpolation by calculating $\mathbb{E}[(S-4)_+]$ and $\mathbb{E}[(S-6)_+]$.

Solution:

We compute $\mathbb{E}[S \wedge 2]$ and $\mathbb{E}[S \wedge 4]$ using Theorem 6.1:

$$
\begin{aligned}
\mathbb{E}[S \wedge 4] &= 2\left[\mathbb{P}(S>0) + \mathbb{P}(S>2)\right] = 2(0.8 + 0.728) = 3.056, \\
\mathbb{E}[S \wedge 6] &= 2\left[\mathbb{P}(S>0) + \mathbb{P}(S>2) + \mathbb{P}(S>4)\right] = 2(0.8 + 0.728 + 0.66208) = 4.38016.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}[(S-4)_+] &= \mathbb{E}[S] - \mathbb{E}[S \wedge 4] = 12.544, \\
\mathbb{E}[(S-6)_+] &= \mathbb{E}[S] - \mathbb{E}[S \wedge 6] = 11.219844.
\end{aligned}
$$

By interpolation, we have

$$
\mathbb{E}[(S-5)_+] = \frac{5-4}{6-4}\mathbb{E}[(S-6)_+] + \frac{6-5}{6-4}\mathbb{E}[(S-4)_+] = 11.88192.
$$